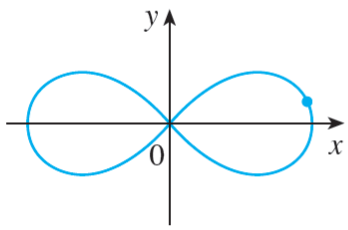
Exercise 31

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$2(x^2 + y^2)^2 = 25(x^2 - y^2), \quad (3, 1), \quad \text{(lemniscate)}$$



Solution

The aim is to evaluate y' at $x = 3$ and $y = 1$ in order to find the slope there. Differentiate both sides of the given equation with respect to x.

$$\frac{d}{dx}[2(x^2 + y^2)^2] = \frac{d}{dx}[25(x^2 - y^2)]$$

$$2 \cdot 2(x^2 + y^2) \cdot \frac{d}{dx}(x^2 + y^2) = 25\left[\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2)\right]$$

$$2 \cdot 2(x^2 + y^2) \cdot [(2x) + (2yy')] = 25[(2x) - (2yy')]$$

$$8(x^2 + y^2)(x + yy') = 50(x - yy')$$

$$4(x^2 + y^2)(x + yy') = 25(x - yy')$$

Solve for y'.

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 25x - 25yy'$$

$$(4x^2y + 4y^3 + 25y)y' = 25x - 4x^3 - 4xy^2$$

$$y' = \frac{25x - 4x^3 - 4xy^2}{4x^2y + 4y^3 + 25y}$$

Evaluate y' at $x = 3$ and $y = 1$.

$$y'(3, 1) = \frac{25(3) - 4(3)^3 - 4(3)(1)^2}{4(3)^2(1) + 4(1)^3 + 25(1)} = -\frac{9}{13}$$

Therefore, the equation of the tangent line to the curve represented by $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at $(3, 1)$ is

$$y - 1 = -\frac{9}{13}(x - 3).$$

Below is a graph of the curve and the tangent line at $(3, 1)$.